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In this supplement we discuss the relation between the integral and differential forms of Maxwell's equations, derive the 3d wave equation for vacuum electromagnetic fields, find the general form of a plane wave solution, and discuss the field energy conservation theorem. The second section summarizes a few mathematical items from vector calculus needed for this discussion, including the continuity equation.

1 Maxwell's equations

Maxwell's equations in differential form are the following equations:

$$
\begin{aligned}
\nabla\cdot\mathbf{E} &= \rho/\epsilon_0 && \textit{Gauss' law (electric)} && (1)\\
\nabla\cdot\mathbf{B} &= 0 && \textit{Gauss' law (magnetic)} && (2)\\
\nabla\times\mathbf{E} &= -\partial_t\mathbf{B} && \textit{Faraday's law} && (3)\\
\nabla\times\mathbf{B} &= \mu_0(\mathbf{j}+\epsilon_0\,\partial_t\,\mathbf{E}) && \textit{Ampère-Maxwell law} && (4)
\end{aligned}
$$

The first two equations are scalar equations, while the last two are vector equations. The notation ∂_t stands for $\partial/\partial t$. We now discuss how these differential equations are obtained from the integral equations.

1.1 Gauss' law

The integral form of Gauss' law is $\int_{\partial V}\mathbf{E}\cdot\mathbf{dS} = q/\epsilon_0$. This can be written as an equality between three dimensional volume integrals, by writing the total charge enclosed q as the integral of the charge density over the volume, and using the divergence theorem (21) to express the flux integral as the volume integral of the divergence of $\mathbf{E}$. Since the two volume integrals are equal for *any* integration volume V, we can equate the integrands, thus obtaining the differential form of Gauss' law (1). For the magnetic field there are no magnetic monopole charges, hence we have simply (2).

1.2 Faraday's law

Faraday's law states that the loop integral of the induced electric field is minus the time rate of change of the magnetic flux through the loop, $\int_{\partial S}\mathbf{E}\cdot\mathbf{dl} = -(d/dt)\int_S\mathbf{B}\cdot\mathbf{dS}$. We can use Stoke's theorem (20) to write the loop integral of $\mathbf{E}$ as a surface integral of the curl of $\mathbf{E}$. Equating integrands then yields the differential form of Faraday's law, eqn. (3).

The divergence of the left hand side of Faraday's law, $\nabla\cdot(\nabla\times\mathbf{E})$, vanishes identically (see (16)), so if Faraday's law is consistent it must be true that $\nabla\cdot\partial_t\,\mathbf{B}$ also vanishes. Since the time and space partial derivatives commute, this is the same as $\partial_t\,\nabla\cdot\mathbf{B}$, which vanishes thanks to the magnetic version of Gauss' law (2). So the absense of magnetic charges is required for Faraday's law to be self-consistent.

1.3 Ampère's law

Ampère's law states that the loop integral of the magnetic field is μ_0 times the current flowing though the loop, $\int_{\partial S}\mathbf{B}\cdot\mathbf{dl} = \mu_0\,i$. We can use Stoke's theorem (20) to write the loop integral of

$\mathbf{B}$ as a surface integral of the curl of $\mathbf{B}$, and we can write the current i though the loop as the flux integral of the *current density* $\mathbf{j}$ though a surface spanning the loop, $i = \int_S \mathbf{j} \cdot \mathbf{dS}$. Equating integrands we obtain the field equation $\nabla \times \mathbf{B} = \mu_0 \, \mathbf{j}$.

Now if we check for self-consistency we discover a problem: the divergence of the left hand side vanishes identically, but the divergence of the right hand side, $\nabla \cdot \mathbf{j}$, does not. In fact, charge conservation implies the continuity equation (23), i.e., $\nabla \cdot \mathbf{j} = -\partial_t \, \rho$. In a steady state situation, where all time derivatives vanish, Ampère's law is self-consistent as it should be, since it was after all discovered to apply in steady state situations. However in the presence of time dependent charge densities it cannot be correct.

To ensure consistency of Ampère's law in a time dependent setting we must add something to the right hand side so that it will be divergence free. To see what should be added, note that $\nabla \cdot \mathbf{j} = -\partial_t \, \rho = -\epsilon_0 \, \partial_t \, \nabla \cdot \mathbf{E} = -\nabla \cdot (\epsilon_0 \, \partial_t \, \mathbf{E})$, where Gauss' law (1) was used in the second equality. Thus Gauss' law and the continuity equation imply that the vector $\mathbf{j} + \epsilon_0 \, \partial_t \, \mathbf{E}$ has vanishing divergence, hence is a suitable source term for the curl of the magnetic field. The resulting modified equation, (4), might be called the *Ampère-Maxwell* equation. The extra term, $\epsilon_0 \, \partial_t \, \mathbf{E}$, is called the *displacement current* density.

1.4 Electromagnetic wave equation

Maxwell's equations are *first order, coupled* partial differential equations for $\mathbf{E}$ and $\mathbf{B}$. They can be uncoupled by taking another derivative. In vacuum, i.e. with vanishing charge and current densities, we have

$$\begin{aligned} \partial_t^2 \, \mathbf{E} &= \partial_t \, \frac{1}{\epsilon_0 \mu_0} \nabla \times \mathbf{B} && (5) \\ &= -\frac{1}{\epsilon_0 \mu_0} \nabla \times (\nabla \times \mathbf{E}) && (6) \\ &= \frac{1}{\epsilon_0 \mu_0} \nabla^2 \mathbf{E}. && (7) \end{aligned}$$

In the first line we used the Ampere-Maxwell law (4), in the second line we used Faraday's law (3) together with the fact that partial derivatives commute, and in the last line we used Gauss' law (1), together with the identity (17). An identical equation results for $\mathbf{B}$ as well. Equation (7) is the three dimensional wave equation for each component of the electric field, with a wave speed $c = 1/\sqrt{\mu_0 \epsilon_0} = 3 \times 10^8$ m/s, the speed of light.

1.5 Plane wave solutions

Rather than messing around with the wave equation in general, let's go back to Maxwell's equations and find the plane wave solutions directly. We choose coordinates so the propagation direction is z, so that $\mathbf{E} = \mathbf{E}(z,t)$ and $\mathbf{B} = \mathbf{B}(z,t)$. Gauss' law implies then that $\partial_z \, E^z = 0$, so $E^z = E^z(t)$, and similarly $B^z = B^z(t)$. This simplifies the curl, which becomes $\nabla \times \mathbf{E} = (-\partial_z \, E^y, \partial_z \, E^x, 0)$, and similarly $\nabla \times \mathbf{B} = (-\partial_z \, B^y, \partial_z \, B^x, 0)$. The z-component of Faraday's law thus implies that in fact B^z can have no time dependence either, so it must be strictly constant in space and time. Similarly, the z-component of the Ampère-Maxwell law implies that E^z is constant. These constant z-components have nothing to do with a wave, so in the wave itself the electric and magnetic fields are *transverse*, i.e. perpendicular to the propagation direction.

The x-and y components of Faraday's law yield

$$\partial_t B^x = \partial_z E^y \qquad \partial_t B^y = -\partial_z E^x \tag{8}$$

while those of the Ampère-Maxwell law yield

$$\partial_t E^x = \frac{1}{\epsilon_0\mu_0}\partial_z B^y \qquad \partial_t E^y = \frac{1}{\epsilon_0\mu_0}\partial_z E^x \tag{9}$$

Taking another time derivative we can decouple these equations to obtain

$$\partial_t^2 E^x = \frac{1}{\epsilon_0\mu_0}\partial_z^2 E^x, \tag{10}$$

and similarly for E^y, B^x, and B^y. Equation (10) is the one-dimensional wave equation for the scalar field E^x, with wave speed $c = 1/\sqrt{\epsilon_0\mu_0}$. The general solution to this equation is a superposition of waves propagating in the $+\hat{z}$ and $-\hat{z}$ directions, $E^x(z,t) = f(z-ct) + g(z+ct)$.

Let us now further specialize to the case where our plane wave is propagating purely in the $+\hat{z}$ direction. Then all the time derivatives are $-c$ times the space derivatives. In this case, Faraday's law (8) becomes

$$-c\partial_z B^x = \partial_z E^y \qquad c\partial_z B^y = \partial_z E^x, \tag{11}$$

and the Ampère-Maxwell law is redundant with this. It follows that, up to an additive constant, $cB^x = -E^y$ and $cB^y = E^x$. Hence, apart from any constantfields present, the electric and magnetic wave fields are perpendicular to the propagation direction, perpendicular to each other, and $|\mathbf{E}| = c|\mathbf{B}|$. Moreover the propagation direction coincides with the direction of $\mathbf{E}\times\mathbf{B}$.

The energy density stored in an electromagnetic field turns out to be

$$u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2. \tag{12}$$

In a traveling plane wave $|\mathbf{E}| = c|\mathbf{B}|$, so the electric and magnetic field energy densities are equal, the total being given by $u = |\mathbf{E}||\mathbf{B}|/\mu_0 c$. The flux of energy per unit area per unit time is then given by $cu = |\mathbf{E}||\mathbf{B}|/\mu_0$. Since the flux is in the direction of $\mathbf{E}\times\mathbf{B}$, we see that the energy flux vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}. \tag{13}$$

We'll now see that this vector, called the *Poynting vector*, represents the energy flux in all situations, not just in a plane wave.

1.6 Conservation of field + particle energy

A straightforward calculation using Faraday's law and the Ampère-Maxwell law, as well as the product rule identity (18), yields for the time derivative of the field energy density:

$$\partial_t\, u = -(\nabla\cdot\mathbf{S} + \mathbf{E}\cdot\mathbf{j}). \tag{14}$$

This equation expresses the energy conservation principle for electromagnetic fields and charges. It has the form of the continuity equation (23) with an extra term $\mathbf{E}\cdot\mathbf{j}$). It states that the rate of change of the field energy in a region is the negative of the rate at which field energy is flowing out of the region minus the rate that the field is doing work on the charges in that region. That the last term represents this work is seen by writing the current density as the charge density times the velocity of the charge, $\mathbf{j} = \rho\mathbf{v}$, so that $\mathbf{E}\cdot\mathbf{j} = \rho\mathbf{E}\cdot\mathbf{v} = (\mathbf{F}\cdot\mathbf{v})/(\text{volume})$.

2 Vector calculus

2.1 Second derivatives

The fact that mixed partial derivatives commute implies a couple of important identities for any scalar function f and vector field $\mathbf{E}$:

$$\nabla \times (\nabla f) = 0 \tag{15}$$
$$\nabla \cdot (\nabla \times \mathbf{E}) = 0. \tag{16}$$

Another useful identity is for the curl of the curl:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \tag{17}$$

where $\nabla^2 = \nabla \cdot \nabla$ is the *Laplacian.*

2.2 Product rule

There are many, but the one we need here is for the divergence of a cross-product:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = (\nabla \times \mathbf{E}) \cdot \mathbf{B} - \mathbf{E} \cdot (\nabla \times \mathbf{B}). \tag{18}$$

2.3 Stokes' theorem and the divergence theorem

We need the two and three dimensional versions of the fundamental theorem of calculus, the so-called Stokes and divergence theorems:

$$\int_a^b \nabla f \cdot \mathbf{dl} = f(b) - f(a) \tag{19}$$
$$\int_S \nabla \times \mathbf{E} \cdot \mathbf{dS} = \int_{\partial S} \mathbf{E} \cdot \mathbf{dl} \tag{20}$$
$$\int_V \nabla \cdot \mathbf{j} \, dV = \int_{\partial V} \mathbf{j} \cdot \mathbf{dS}. \tag{21}$$

In each case, the left hand side is an integral over a region of dimension 1, 2, and 3, respectively, while the right hand side is an integral (or sum, in the case of eqn (19)) over the boundary of that region, which has one fewer dimension.

2.4 Conservation or continuity equation

Suppose the current $\mathbf{j}$ describes the flux of some quantity (for example charge or mass or energy). Then $\mathbf{j} \cdot \mathbf{dS}$ is the rate of flow of that quantity through the area element $\mathbf{dS}$ per unit time, so the rate of flow out of a volume V is $\int_{\partial V} \mathbf{j} \cdot \mathbf{dS}$ (with the area element proportional to the *outward* normal). The divergence theorem (21) shows that this is the same as $\int_V \nabla \cdot \mathbf{j} \, dV$.

Now let ρ denote the density of the quantity whose current is $\mathbf{j}$, so $\rho \, dV$ is the amount of the quantity in the volume element dV, and $\int_V \rho \, dV$ is the amount in the finite volume V. If the quantity is conserved, then the only reason the quantity in V can change is if some of it flows in or out of V, across the bounding surface ∂V. This is expressed by the integral conservation equation,

$$\frac{d}{dt} \int_V \rho \, dV = - \int_V \nabla \cdot \mathbf{j} \, dV, \tag{22}$$

or, equating integrands,

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 \tag{23}$$

(where ∂_t stands for $\partial / \partial t$). This is also called the *continuity equation.*